Technical Report on the Evaluation of Median Rank Regression and Maximum Likelihood Estimation Techniques for a Two-Parameter Weibull Distribution

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Abstract

Practitioners frequently model failure times in reliability analysis via the Weibull distribution. Often risk managers must make decisions after only a few failures. Thus, an important question is how to estimate the parameters of this distribution for small sample sizes. This study evaluates two methods: maximum likelihood estimation and median rank regression. Asymptotically, we know that maximum likelihood estimation has superior properties; however, this study seeks to evaluate these two methods for small numbers of failures and high degrees of censoring. Specifically, this paper compares the two estimation methods based on their ability to estimate the individual parameters, and the methods' ability to predict future failures. The last section of the paper provides recommendations on which method to use based on sample size, the parameter values, and the degree of censoring present in the data.

1. Simulation Methodology

This study uses several quantities to evaluate how well the two methods estimate the true values of β and η. Some notation will be useful in defining the evaluation criteria. Let,

- $\hat{\beta}_j$ is the estimate for β for each of the 1000 estimates from the first simulation step
- $\hat{\eta}_j$ is the estimate for η for each of the 1000 estimates from the first simulation step
- $\tilde{\hat{\beta}}_i$ is the median for each of the 100 second simulation steps
- $\tilde{\hat{\eta}}_i$ is the median for each of the 100 second simulation steps.

In addition to the β and η estimates, we are introducing a measure of variability, the median absolute deviation (MAD) for the estimation of each of the parameters. We follow the same simulation strategy to get a final average estimate of the MAD as we do for the actual β and η. At each of the 1000 replications of the first simulation step we calculate the absolute value of the difference between the parameter estimate obtained with each of MLE and MRR and their medians over these 1000 replications. Then, at the second simulation step, we take the median of

the 1000 absolute values. At the end of the second simulation step, we have four MAD vectors corresponding to the two parameters estimation with each of the two methods: MLE and MRR. MAD is a robust (less sensitive to the assumption of normality) estimator of variability for non-normal distribution (Hoaglin (1983, page 291-293)). We can obtain an estimate of the standard deviation by multiply MAD by a constant *k*. A commonly used value for *k* is 1.4826 (Hoaglin, 1983).

We also calculate a mean square error-type measure for each of the two parameters estimation with each of the two methods. This measure includes a squared-bias component and a variance component. We calculate a bias for each of the 1000 estimates of the two parameters for each of the MLE and MRR methods as the difference between the estimate and the true value for the parameter. Then we store the median bias from these 1000 values and repeat the process 100 times at the second step of the simulation. The squared median bias is the first component of the pseudo-MSE measure. The second component is an approximate variance calculated as the squared standard deviation estimated from the MAD. We calculate this measure for each of the 100 replications of the second stage simulation. We introduce the MAD measure as a comparison criteria in terms of the variance of the utilized method, while the pseudo-MSE is intended to be an indication of both accuracy and precision of the method.

The product of the two-stage simulation consists of vectors containing 100 elements for each of: the median parameter estimates, the MAD estimates and the pseudo-MSE measures for each parameter of interest, calculated using both MLE and MRR measures. For each of these measures we calculate the average and the simulation error.

Then the evaluation criteria are:

1. The overall estimate for β: $\bar{\bar{\beta}} = \frac{\sum_{i=1}^{100} \tilde{\beta}_i}{100} = \frac{\sum_{i=1}^{100} Med\left(\hat{\beta}\right)_i}{100}$ (the average median)
2. The overall estimate for η: $\bar{\bar{\eta}} = \frac{\sum_{i=1}^{100} \tilde{\eta}_i}{100} = \frac{\sum_{i=1}^{100} Med(\hat{\eta})_i}{100}$ (the average median)
3. The bias of $\bar{\bar{\beta}}$ and $\bar{\bar{\eta}}$: $Bias(\bar{\beta}) = \bar{\beta} - \beta$ and $Bias(\bar{\eta}) = \bar{\eta} - \eta$
4. The variance of $\bar{\bar{\beta}}$ and $\bar{\bar{\eta}}$, used to calculate the simulation error: $Var(\bar{\beta}) = \frac{\sum_{i=1}^{100}(\tilde{\beta}_i - \bar{\beta})^2}{99}$ and $Var(\bar{\eta}) = \frac{\sum_{i=1}^{100}(\tilde{\eta}_i - \bar{\eta})^2}{99}$
5. The simulation error: $SimError(\bar{\beta}) = \sqrt{\frac{Var(\bar{\beta})}{100}}$ and $SimError(\bar{\eta}) = \sqrt{\frac{Var(\bar{\eta})}{100}}$

6. The average Median Absolute Deviation (MAD) of β: $\overline{MAD_\beta} = \frac{\sum_{i=1}^{100} Med\left(\left|\widehat{\beta} - \widetilde{\beta}\right|\right)_i}{100}$

7. The average Median Absolute Deviation (MAD) of η: $\overline{MAD_\eta} = \frac{\sum_{i=1}^{100} Med\left(\left|\widehat{\eta} - \widetilde{\eta}\right|\right)_i}{100}$

8. The average pseudo-MSE of β:

$$\overline{pseudo - MSE_\beta} = \frac{\sum_{i=1}^{100}\left(\left(Med\left(\left|\widehat{\beta} - \beta\right|\right)\right)^2 + \left(k * MAD_\beta\right)^2\right)_i}{100}$$

9. The average pseudo-MSE of η:

$$\overline{pseudo - MSE_\eta} = \frac{\sum_{i=1}^{100}\left(\left(Med\left(\left|\widehat{\eta} - \eta\right|\right)\right)^2 + \left(k * MAD_\eta\right)^2\right)_i}{100}$$

- Note that the simulation error is equal here to the standard error of the median, for each quantity of interest.
- We calculate a simulation error for each of the average median estimate, MAD, and pseudo-MSE criteria.

Asymptotic theory tells us that MLE will outperform MRR for large sample sizes. We are investigating small sample sizes (n<10) because of their practical importance to practitioners. We also evaluate one larger sample size (n=25) in order to verify the expected trend of the two methods' performance. It is anticipated that as the sample size increases MLE will start to dominate MRR. A key question is at what sample size does MLE dominate MRR and does it change for different values of the Weibull parameters.

Table 1: Parameter Values for Simulation Study

β	Mean	η
0.5	1000	333.33
1.0	1000	1000
3.0	1000	1119.95

5.0	1000	1089.13

2. The Uncensored Case

Data is generated in the uncensored data case using the values of β and η given in This study uses several quantities to evaluate how well the two methods estimate the true values of β and η. Some notation will be useful in defining the evaluation criteria. Let,

- $\hat{\beta}_j$ is the estimate for β for each of the 1000 estimates from the first simulation step
- $\hat{\eta}_j$ is the estimate for η for each of the 1000 estimates from the first simulation step
- $\tilde{\hat{\beta}}_i$ is the median for each of the 100 second simulation steps
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In addition to the β and η estimates, we are introducing a measure of variability, the median absolute deviation (MAD) for the estimation of each of the parameters. We follow the same simulation strategy to get a final average estimate of the MAD as we do for the actual β and η. At each of the 1000 replications of the first simulation step we calculate the absolute value of the difference between the parameter estimate obtained with each of MLE and MRR and their medians over these 1000 replications. Then, at the second simulation step, we take the median of the 1000 absolute values. At the end of the second simulation step, we have four MAD vectors corresponding to the two parameters estimation with each of the two methods: MLE and MRR. MAD is a robust (less sensitive to the assumption of normality) estimator of variability for non-normal distribution (Hoaglin (1983, page 291-293)). We can obtain an estimate of the standard deviation by multiply MAD by a constant *k*. A commonly used value for *k* is 1.4826 (Hoaglin, 1983).

We also calculate a mean square error-type measure for each of the two parameters estimation with each of the two methods. This measure includes a squared-bias component and a variance component. We calculate a bias for each of the 1000 estimates of the two parameters for each of the MLE and MRR methods as the difference between the estimate and the true value for the parameter. Then we store the median bias from these 1000 values and repeat the process 100 times at the second step of the simulation. The squared median bias is the first component of the pseudo-MSE measure. The second component is an approximate variance calculated as the squared standard deviation estimated from the MAD. We calculate this measure for each of the 100 replications of the second stage simulation. We introduce the MAD measure as a comparison criteria in terms of the variance of the utilized method, while the pseudo-MSE is intended to be an indication of both accuracy and precision of the method.

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Then the evaluation criteria are:

10. The overall estimate for β: $\bar{\tilde{\beta}} = \frac{\sum_{i=1}^{100} \tilde{\beta}_i}{100} = \frac{\sum_{i=1}^{100} Med\left(\hat{\beta}\right)_i}{100}$ (the average median)
11. The overall estimate for η: $\bar{\tilde{\eta}} = \frac{\sum_{i=1}^{100} \tilde{\eta}_i}{100} = \frac{\sum_{i=1}^{100} Med(\hat{\eta})_i}{100}$ (the average median)
12. The bias of $\bar{\tilde{\beta}}$ and $\bar{\tilde{\eta}}$: $Bias(\bar{\beta}) = \bar{\beta} - \beta$ and $Bias(\bar{\eta}) = \bar{\eta} - \eta$
13. The variance of $\bar{\tilde{\beta}}$ and $\bar{\tilde{\eta}}$, used to calculate the simulation error: $Var(\bar{\beta}) = \frac{\sum_{i=1}^{100}(\tilde{\beta}_i - \bar{\beta})^2}{99}$ and $Var(\bar{\eta}) = \frac{\sum_{i=1}^{100}(\tilde{\eta}_i - \bar{\eta})^2}{99}$
14. The simulation error: $SimError(\bar{\beta}) = \sqrt{\frac{Var(\bar{\beta})}{100}}$ and $SimError(\bar{\eta}) = \sqrt{\frac{Var(\bar{\eta})}{100}}$
15. The average Median Absolute Deviation (MAD) of β: $\overline{MAD}_\beta = \frac{\sum_{i=1}^{100} Med\left(\left|\hat{\beta} - \bar{\tilde{\beta}}\right|\right)_i}{100}$
16. The average Median Absolute Deviation (MAD) of η: $\overline{MAD}_\eta = \frac{\sum_{i=1}^{100} Med\left(\left|\hat{\eta} - \bar{\tilde{\eta}}\right|\right)_i}{100}$
17. The average pseudo-MSE of β:

$$\overline{pseudo - MSE}_\beta = \frac{\sum_{i=1}^{100}\left(\left(Med\left(\left|\hat{\beta} - \beta\right|\right)\right)^2 + \left(k * MAD_\beta\right)^2\right)_i}{100}$$

18. The average pseudo-MSE of η:

$$\overline{pseudo - MSE_{\eta}} = \frac{\sum_{i=1}^{100}\left(\left(Med\left(\left|\hat{\eta} - \eta\right|\right)\right)^2 + \left(k * MAD_{\eta}\right)^2\right)_i}{100}$$

- Note that the simulation error is equal here to the standard error of the median, for each quantity of interest.
- We calculate a simulation error for each of the average median estimate, MAD, and pseudo-MSE criteria.

Asymptotic theory tells us that MLE will outperform MRR for large sample sizes. We are investigating small sample sizes (n<10) because of their practical importance to practitioners. We also evaluate one larger sample size (n=25) in order to verify the expected trend of the two methods' performance. It is anticipated that as the sample size increases MLE will start to dominate MRR. A key question is at what sample size does MLE dominate MRR and does it change for different values of the Weibull parameters.

and for sample sizes of 3, 5, 10 and 25.

Table 2: Average Median Estimates and Simulation Errors for β = 0.5 and η=333.3

	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	0.75458	0.0017	0.29221	0.00134	0.2583	0.00233
5	0.62147	0.00099	0.16885	0.00061	0.0875	0.00061
10	0.5523	0.00059	0.09822	0.00035	0.0303	0.0002
25	0.51958	0.00034	0.05631	0.00024	0.01	0.00008
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	0.52058	0.00129	0.20702	0.00095	0.134	0.0011
5	0.50149	0.00098	0.14717	0.00059	0.0692	0.0005
10	0.49333	0.00062	0.1011	0.00037	0.0331	0.00023
25	0.49191	0.00037	0.06442	0.00026	0.0135	0.0001

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	284.36	1.42408	200.772	1.10731	140976	1164.02
5	308.88	0.98655	178.755	0.72723	106072	733.59
10	321.85	0.81588	138.519	0.53971	62443	441.07
25	329.78	0.58981	92.073	0.3624	27297	207.89
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR

3	305.24	1.66953	239.444	1.4544	192743	1792.84
5	307.51	1.24795	204.104	0.87072	139015	947.19
10	312.74	0.97001	158.606	0.59025	83202	532.91
25	322.54	0.62493	108.301	0.4616	38157	300.98

Table 3: Average Median Estimates and Simulation Errors for β=1 and η=1000

	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	1.51655	0.0033	0.51655	0.0033	0.8568	0.01104
5	1.24139	0.00195	0.24139	0.00195	0.1876	0.00304
10	1.10733	0.00123	0.10733	0.00123	0.0373	0.00086
25	1.03897	0.00059	0.03897	0.00059	0.005	0.00015
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	1.05023	0.00239	0.41924	0.00187	0.5472	0.00445
5	1.00524	0.00175	0.29429	0.00113	0.2766	0.00197
10	0.98859	0.00127	0.2032	0.00073	0.1332	0.00091
25	0.983	0.00074	0.12996	0.00051	0.0549	0.00041

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	973.03	2.22289	392.956	1.62145	499904	3849.92
5	991.64	1.9121	305.71	1.19117	299961	2255.6
10	1011.84	1.09312	218.816	0.83485	152914	1139.37
25	1013.54	0.82381	139.123	0.47471	61864	417.88
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	962.81	2.6346	444.878	1.75964	643430	4735.08
5	958.8	2.07319	352.762	1.2579	405529	2701.77
10	972.45	1.46199	259.046	1.05223	217652	1684.83
25	980.26	0.98223	167.013	0.61471	90015	632.72

Table 4: Average Median Estimates and Simulation Errors for β=3 and η=1119.95

	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	4.55524	0.01051	1.76578	0.0088	9.451	0.09313
5	3.73099	0.00617	1.01793	0.00435	3.1798	0.02574
10	3.31366	0.0033	0.59181	0.00199	1.0992	0.00673
25	3.11535	0.00171	0.33899	0.00125	0.3632	0.00259
	MRR					

NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	3.134	0.00692	1.2466	0.00588	4.849	0.04172
5	3.01745	0.00486	0.89161	0.00392	2.536	0.02074
10	2.95939	0.00327	0.60897	0.00206	1.1979	0.00769
25	2.9505	0.0022	0.38966	0.0015	0.4924	0.00367

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1126.72	0.96002	185.566	0.80501	110086	935.73
5	1109.05	0.69353	122.68	0.48283	48317	379.22
10	1111.58	0.50803	83.696	0.33048	22505	174.31
25	1116.89	0.30899	53.043	0.17964	9015	59.73
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1104.12	0.95349	175.813	0.67154	99235	739.08
5	1104.42	0.75629	139.64	0.52944	62639	465.27
10	1108	0.55573	99.767	0.39056	32023	244.92
25	1113.17	0.36893	63.859	0.22364	13111	91.13

Table 5: Average Median Estimates and Simulation Errors for β=5 and η=1089.13

	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	7.53253	0.01665	2.92266	0.01192	25.7449	0.21092
5	6.20809	0.01012	1.6843	0.0066	8.7127	0.06599
10	5.52679	0.0064	0.99431	0.00385	3.0973	0.02253
25	5.19248	0.00339	0.56111	0.00176	0.9985	0.00604
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	5.18656	0.01163	2.06728	0.00879	13.4027	0.10303
5	5.02879	0.00971	1.48351	0.00574	7.0209	0.04863
10	4.93164	0.00614	1.02045	0.00371	3.3641	0.02321
25	4.92134	0.00378	0.64486	0.00235	1.3476	0.00894

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1135.59	0.87604	174.427	1.05668	91128	1004.99
5	1100.28	0.45253	86.28	0.31821	23654	172.63
10	1086.24	0.31296	50.669	0.19423	8247	62.82
25	1087.48	0.17985	30.81	0.12209	3039	23.85

	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1079.92	0.55372	103.646	0.3428	34330	221.4
5	1081.44	0.46937	81.4	0.29102	21225	151.62
10	1082.3	0.35383	58.442	0.216	10956	79.66
25	1085.63	0.22797	37.181	0.1462	4429	34.57

3. The Right-Censored Case

The following table summarizes the total number of units in the sample considered in this study. In right censored data not all of the units in a sample fail.

Table 6: Sample Sizes for Simulation Study

β	Sample Size
0.5	100
	1000
	5000
1.0	100
	1000
	5000
3.0	100
	1000
	5000
5.0	100
	1000
	5000

Table 7: Average Median Estimates and Simulation Errors for β = 0.5, η=333 and Sample Size = 100

	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	0.55854	0.00134	0.21329	0.00103	0.13757	0.00123
5	0.53409	0.00102	0.15771	0.00068	0.07699	0.00062
10	0.51519	0.00065	0.10772	0.00045	0.0366	0.00029
25	0.50543	0.00038	0.06568	0.00023	0.01376	0.00009
	MRR					
	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	0.55568	0.00147	0.27118	0.00141	0.22264	0.00205
5	0.51925	0.0013	0.19396	0.00075	0.11793	0.0008
10	0.49889	0.00081	0.13282	0.00057	0.05663	0.00045
25	0.49243	0.00048	0.08294	0.00033	0.02228	0.00017

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	240.791	3.5773	234.697	3.55161	229389	3881.65
5	295.278	3.20841	276.607	3.12244	267423	3911.03
10	333.686	2.18228	258.933	1.84393	215025	2312.72
25	344.283	0.86506	141.585	0.66247	63187	538.43
	MRR					
	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	299.865	1.74392	244.083	1.56088	202473	1939.91
5	302.27	1.50452	212.386	1.0755	151947	1165.21
10	309.448	1.11395	167.318	0.7963	93322	725.54
25	319.767	0.68372	114.214	0.48067	42628	334.71

Table 8: Average Median Estimates and Simulation Errors for β = 0.5, η=333.3, and Sample Size = 1000

	Sample size n=1000					
	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	0.56337	0.001622	0.214959	0.00099	0.139812	0.001162
5	0.536896	0.001197	0.159735	0.000709	0.078864	0.000638
10	0.517074	0.00076	0.109249	0.000421	0.037561	0.000275
25	0.506999	0.000526	0.067677	0.000254	0.014572	0.000105
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	0.562832	0.001411	0.27474	0.001338	0.227898	0.002001
5	0.519411	0.000979	0.19393	0.000932	0.118127	0.001041
10	0.498475	0.000581	0.133633	0.000568	0.057333	0.000457
25	0.491881	0.000397	0.085805	0.000338	0.023875	0.000176

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	136.88	3.51076	136.207	3.50991	153858	2306.16
5	209.914	3.73992	208.51	3.73419	208239	3615.4
10	281.289	2.80134	271.57	2.71765	266713	3332.3
25	317.651	1.90441	246.006	1.65544	199851	2021.54
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	301.963	1.69652	245.401	1.38024	203885	1641.98
5	304.049	1.34993	212.833	1.08455	152306	1224.88
10	309.998	1.1026	168.314	0.77746	94166.3	752.236

25	318.655	0.664736	116.124	0.444227	44138.9	306.099

Table 9: Average Median Estimates and Simulation Errors for β = 0.5, η=333.3, and Sample Size = 5000

	Sample size n=5000					
	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	0.560591	0.001373	0.21345	0.000921	0.138138	0.001096
5	0.534591	0.001013	0.158464	0.000645	0.077863	0.0006
10	0.516874	0.000701	0.109584	0.000449	0.03776	0.000291
25	0.506775	0.000462	0.068261	0.000251	0.014809	0.000106
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	0.558089	0.00177	0.271543	0.001166	0.223355	0.001693
5	0.51853	0.001156	0.194629	0.000792	0.1191	0.000905
10	0.498785	0.000856	0.134778	0.000546	0.058316	0.00044
25	0.491612	0.000526	0.086047	0.000329	0.023956	0.000174

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	105.667	3.60724	105.261	3.60716	137982	1945.7
5	179.642	4.21526	179.259	4.21472	185249	3549.22
10	253.857	4.46907	251.912	4.45016	252848	5368.85
25	304.571	2.79924	276.075	2.5941	259452	3353.63
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	300.472	1.75122	244.1	1.41762	202155	1741.59
5	301.739	1.41827	211.694	1.03469	151542	1146.96
10	308.126	1.1896	167.505	0.777961	93586.5	720.456
25	318.535	0.80653	116.521	0.468123	44481.2	331.242

Table 10: Average Median Estimates and Simulation Errors for β = 1, η=1000, and Sample Size = 100

	Sample size n=100					
	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	1.11708	0.002674	0.117075	0.002674	0.046099	0.002117
5	1.06473	0.00207	0.064729	0.00207	0.014756	0.000921
10	1.02916	0.001364	0.029164	0.001364	0.003309	0.000285
25	1.00861	0.000576	0.008609	0.000576	0.000342	4.04E-05
	MRR					

NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	1.11135	0.002689	0.542368	0.00268	0.890574	0.002203
5	1.03472	0.002089	0.384075	0.002071	0.464174	0.000936
10	0.995131	0.001364	0.263138	0.001364	0.222439	0.00029
25	0.984739	0.000579	0.165576	0.000581	0.088763	8.92E-05

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	664.327	2.92321	449.867	1.77737	802796	4030.28
5	756.864	2.29854	379.726	1.33549	517905	3059.44
10	851.188	1.84602	265.411	1.0121	241438	1596.77
25	940.055	0.89733	151.546	0.521285	76131.9	486.801
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	948.077	2.7572	459.772	1.98339	693312	5371.82
5	951.957	2.33571	375.309	1.35839	460514	3080.03
10	962.884	1.3369	276.014	1.14226	248148	1954.36
25	979.736	1.01783	180.808	0.72912	105695	827.667

Table 11: Average Median Estimates and Simulation Errors for β = 1, η=1000, and Sample Size = 1000

	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	1.12512	0.00265	0.12512	0.00265	0.05229	0.00216
5	1.07169	0.00202	0.07169	0.00202	0.01773	0.00094
10	1.03704	0.00124	0.03704	0.00124	0.00487	0.00029
25	1.01398	0.00079	0.01398	0.00079	0.00082	0.00008
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	1.12503	0.00331	0.54827	0.00269	0.90778	0.00799
5	1.03613	0.00258	0.39021	0.00177	0.47856	0.00385
10	1.00008	0.00182	0.26902	0.00106	0.23184	0.00167
25	0.98446	0.0009	0.17301	0.00059	0.09673	0.00064

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	516.744	4.0942	458.307	3.13414	999795	5772.83
5	627.623	3.43563	457.27	1.77884	789070	3817.92
10	739.844	2.3563	360.91	1.14449	441675	2561.91
25	852.061	1.23067	223.226	0.86144	161974	1066.47

	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	950.619	2.63328	460.844	1.61239	694483	4408.59
5	951.544	2.07995	375.305	1.3093	460956	2914.88
10	965.642	1.62928	279.395	1.02241	253234	1727.44
25	978.68	1.00733	182.772	0.71823	108214	802.75

Table 12: Average Median Estimates and Simulation Errors for β = 1, η=1000, and Sample Size = 5000

	Sample size n=5000					
	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	1.12118	0.002746	0.121181	0.002746	0.049351	0.002176
5	1.06918	0.002026	0.069181	0.002026	0.016606	0.000952
10	1.03376	0.0014	0.033757	0.0014	0.004265	0.000303
25	1.01394	0.000844	0.013936	0.000844	0.000847	8.90E-05
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	1.11618	0.002779	0.543086	0.002756	0.89342	0.00227
5	1.03706	0.002057	0.389258	0.002029	0.476402	0.000984
10	0.997569	0.001405	0.269555	0.001402	0.233263	0.000324
25	0.983224	0.000846	0.172094	0.000846	0.095822	0.000103

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	445.429	4.42115	426.104	3.9503	1.03E+06	5930.76
5	571.5	3.78685	478.067	2.244	902872	4064.02
10	699.785	2.68788	409.242	1.29699	558070	3362.13
25	818.156	1.56752	267.517	1.09518	225412	1794.86
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	949.032	2.77202	459.273	1.86206	691977	4990.79
5	951.175	2.22398	376.057	1.464	462953	3398.12
10	961.273	1.85401	278.265	1.06226	252048	1825.35
25	977.478	1.23734	183.577	0.706503	109162	813.805

Table 13: Average Median Estimates and Simulation Errors for β = 3, η=1119.95, Sample Size = 100

	Estimation, Sample size n=100,Beta=3					
	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR

3	3.35123	0.00802	1.27971	0.00616	4.95252	0.04425
5	3.20454	0.00611	0.94625	0.00411	2.77155	0.02214
10	3.09113	0.00393	0.64633	0.00268	1.31776	0.0104
25	3.03255	0.00229	0.39406	0.00138	0.49522	0.00341
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	3.33406	0.00884	1.6271	0.00847	8.01516	0.07397
5	3.11551	0.00781	1.16375	0.00449	4.24563	0.02872
10	2.99333	0.00484	0.7969	0.00339	2.03853	0.01613
25	2.9546	0.00287	0.49762	0.00199	0.80201	0.00614

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1164.88	2.73E+00	470.883	2.04073	688962	5018.81
5	1114.85	2.13	316.698	1.50646	322914	2659.31
10	1118.61	1.22E+00	186.848	0.69131	111911	779.518
25	1125.14	4.70E-01	81.1498	0.32454	21064.5	166.702
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1100.12	1.06667	186.601	0.73022	112034	831.878
5	1101.65	0.90609	149.409	0.53396	71710.6	496.809
10	1106.06	0.66493	107.908	0.37762	37404.4	250.504
25	1112.19	0.39479	68.2599	0.25033	14965.4	108.199

Table 14: Average Median Estimates and Simulation Errors for β = 3, η=1119.95, Sample Size = 1000

	Sample size n=1000					
	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	3.37535	0.007954	1.28356	0.005711	4.9936	0.04071
5	3.21617	0.005545	0.95896	0.003961	2.84113	0.021472
10	3.10055	0.003678	0.655029	0.002418	1.35203	0.009305
25	3.04124	0.002496	0.404407	0.001498	0.520139	0.003649
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	1.64481	0.009931	1.64481	0.008083	8.17006	0.071897
5	1.16159	0.007018	1.16159	0.00545	4.23876	0.036239
10	0.800209	0.004811	0.800209	0.003276	2.0545	0.015905
25	0.513613	0.003234	0.513613	0.002044	0.855166	0.006599
	MLE					

NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1130.28	5.22672	665.797	4.59301	1.41E+06	15171.2
5	1069.44	3.47449	493.169	2.59998	808668	6710.37
10	1087.66	1.90423	336.964	1.39441	371718	2756.85
25	1110.44	1.17301	184.172	0.800884	109493	883.308
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1101.11	1.01716	186.385	0.619307	111672	734.95
5	1102.01	0.80729	147.83	0.543358	70243.6	498.23
10	1106.08	0.652515	108.117	0.384281	37576.4	258.771
25	1111.65	0.399882	69.6696	0.264117	15617.2	115.437

Table 15: Average Median Estimates and Simulation Errors for β = 3, η=1119.95, Sample Size = 5000

	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	3.36505	0.00891	1.27797	0.0055	4.95349	0.03974
5	3.20698	0.00521	0.95202	0.00391	2.80649	0.02137
10	3.10524	0.00464	0.66095	0.00262	1.37455	0.01031
25	3.04393	0.00234	0.40937	0.00148	0.53357	0.00384
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	3.35064	0.011	1.63021	0.0074	0.03974	0.06431
5	3.11613	0.01171	1.16674	0.00718	0.02137	0.04618
10	2.99767	0.00595	0.80631	0.00333	0.01031	0.01574
25	2.95328	0.00311	0.51977	0.00214	0.00384	0.00696

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1128.2	6.4654	781.728	5.64881	1953370	21088.8
5	1044.17	3.88427	594.298	3.16577	1189790	9751.8
10	1065.05	3.23811	431.974	2.21642	619729	5202.3
25	1100.67	1.59038	257.546	1.01223	215252	1557.1
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1100.43	1.09092	187.258	0.71925	112886	867.31
5	1101.06	0.88923	148.237	0.53596	70695	494.788
10	1105.3	0.73213	108.81	0.3976	38114	267.315
25	1111.87	0.42091	69.79	0.23213	15631	101.836

Table 16: Average Median Estimates and Simulation Errors for β = 5, η=1119.13, Sample Size = 100

	Sample size n=100					
	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	5.58538	0.013371	2.13285	0.010268	13.757	0.122908
5	5.32364	0.010351	1.56246	0.006121	7.58152	0.05545
10	5.14582	0.006821	1.06719	0.004146	3.59106	0.0266
25	5.04125	0.003208	0.654272	0.002445	1.36484	0.009936
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	5.55677	0.014731	2.71184	0.014112	22.2643	0.205474
5	5.17359	0.011908	1.92037	0.008134	11.6043	0.089548
10	4.97565	0.00781	1.31569	0.004809	5.56097	0.037715
25	4.9237	0.005204	0.827881	0.003016	2.21907	0.015216

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1284.2	2.10854	379.504	2.01087	404832	3987.35
5	1158.75	1.41479	216.169	1.05035	140449	1228.26
10	1100.83	0.789784	112.049	0.487581	39787.7	331.237
25	1093.22	0.2873	47.3679	0.195166	7176.64	58.526
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1077.5	0.626877	109.936	0.414264	38744.9	280.921
5	1078.41	0.529949	87.6362	0.32339	24600.3	176.466
10	1080.91	0.299966	62.9378	0.246503	12736.9	94.8359
25	1084.67	0.225498	40.253	0.168196	5196.95	42.2091

Table 17: Average Median Estimates and Simulation Errors for β = 5, η=1089.13, Sample Size = 1,000

	Sample size n=1000					
	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	5.6337	0.014115	2.14959	0.009899	13.9812	0.116183
5	5.36896	0.00979	1.59735	0.00709	7.88643	0.063779
10	5.17074	0.00581	1.09249	0.00421	3.75614	0.027505
25	5.06999	0.003969	0.67677	0.002537	1.45718	0.010499
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	5.62832	0.016217	2.7474	0.013382	22.7898	255.369

5	5.19411	0.011972	1.9393	0.009324	11.8127	164.934
10	4.98475	0.007596	1.33633	0.005681	5.73331	105.066
25	4.91881	0.005261	0.858045	0.00338	2.38746	35.6311

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1376.98	4.67292	614.291	4.42201	1.03E+06	13564.9
5	1181.28	2.45419	368.083	1.98473	403389	3800.88
10	1094.95	1.105	210.464	0.803615	141206	1028.23
25	1083.72	0.651883	108.838	0.457503	38201	308.156
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1078.27	0.606217	109.881	0.364393	38636.1	255.369
5	1079.07	0.479741	86.961	0.305244	24224.6	164.934
10	1081.19	0.382079	63.4465	0.264419	12924.3	105.066
25	1084.22	0.225697	40.784	0.140277	5341.39	35.6311

Table 18: Average Median Estimates and Simulation Errors for β = 5, η=1089.13, Sample Size = 5,000

	MLE					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	5.60842	0.01485	2.12996	0.00916	13.7597	0.1104
5	5.34497	0.00869	1.5867	0.00652	7.7958	0.05937
10	5.17539	0.00773	1.10157	0.00437	3.8182	0.02863
25	5.07322	0.0039	0.68228	0.00246	1.4822	0.01066
	MRR					
NF	Beta	SimErr	MAD	SimErr	MSE	SimERR
3	5.5844	0.01834	2.71702	0.01233	22.3605	0.17864
5	5.19355	0.01951	1.94457	0.01196	11.8636	0.12828
10	4.99611	0.00991	1.34385	0.00555	5.7942	0.04372
25	4.92214	0.00519	0.86629	0.00356	2.4258	0.01933

	MLE					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR
3	1460.14	5.40134	779.699	4.91817	1625540	18496.9
5	1205.28	3.15182	472.11	2.74883	660311	6662.6
10	1090.36	1.89687	278.205	1.16321	247778	1837.2
25	1078.34	0.93465	153.511	0.60107	76207	563
	MRR					
NF	Eta	SimErr	MAD	SimErr	MSE	SimERR

3	1077.69	0.64115	110.274	0.44556	38974.4	314.502
5	1078.06	0.52211	87.06	0.30884	24315.5	170.06
10	1080.56	0.42954	63.822	0.22616	13085.6	90.176
25	1084.4	0.24633	40.809	0.14049	5344.3	35.578

Another quantity of interest used in risk management is the estimation of the number of cycles before which a certain proportion of population fails. In practice, predicting 1% and 10% percentiles is of interest. Nelson (1985) and Meeker and Escobar (1998) provide the formula for the failure time of the p^{th} percentile:

$$t_p = exp\left[\mu + \Phi_{SEV}^{-1}(p)\sigma\right]$$

where $\mu = log(\eta)$, $\sigma = 1/\beta$ and $\Phi_{SEV}^{-1}(p)$ is the inverse CDF for the smallest extreme value distribution. It can be easily shown that this equation is equivalent to the formula given by Abernethy (2004):

$$R_p = \hat{\eta}\log\left(\frac{1}{1-p}\right)^{1/\hat{\beta}}.$$

The percentile prediction uses the same two-stage simulation process as the parameter prediction because the percentile estimation faces the same skewness issues. For the percentile prediction, the difference between the predicted percentile and the true percentile at the first simulation stage is calculated. Then the median difference is stored from the first 1000 replications. This process is repeated 100 times and the final estimate of the difference between the percentile prediction and the true percentile value is the average of the medians. Therefore, a small magnitude number implies a strong ability of the method to predict a certain percentile, while a large magnitude number implies a weak ability to predict a percentile. A positive associated sign indicates the method's tendency to over predict while a negative sign indicates a tendency to under predict the respective percentile. In addition to these average median differences between estimated percentile and the true percentile, we calculate the MAD and pseudo-MSE and the associated simulation error, in a similar manner as for the parameter estimation study.

Table 19: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 0.5 and η=333, Sample Size n=100

	1st Percentile Differences (Estimated-True)					
	MLE					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	0.025549	0.000153	0.021637	0.000131	0.001688	0.00002
5	0.0188147	0.000139	0.018056	0.000115	0.001075	1.42E-05
10	0.0128547	0.000111	0.018763	0.00012	0.000944	0.000011

25	0.0096938	0.000107	0.02236	0.000159	0.0012	1.64E-05
	MRR					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	0.0532557	0.001826	0.065695	0.001781	0.013343	0.000869
5	0.0239105	0.000931	0.031238	0.00092	0.002987	0.000192
10	0.0120602	0.000342	0.017891	0.000332	0.000885	3.55E-05
25	0.007364	0.000138	0.01443	0.000134	0.000518	1.04E-05

Table 20: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 0.5 and η=333, Sample Size n=1000

	1st Percentile, Sample size n=1000,Beta=0.5					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	0.0104636	0.000207	0.02833	0.000204	0.001887	2.99E-05
5	0.0097863	0.000109	0.017755	9.54E-05	0.000792	9.07E-06
10	0.00506881	2.22E-05	0.003328	1.61E-05	5.01E-05	4.44E-07
25	0.00415842	3.43E-05	0.006445	2.52E-05	0.000109	7.52E-07
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	0.0488671	0.002087	0.111141	0.002064	0.030898	0.001334
5	0.0113901	0.00088	0.066333	0.000988	0.010091	0.000321
10	0.00020257	0.000388	0.047006	0.000429	0.004912	9.27E-05
25	-0.0017129	0.000229	0.032158	0.000242	0.002294	3.49E-05

Table 21: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 0.5 and η=333, Sample Size n=5000

	1st Percentile, Sample size n=5000,Beta=0.5					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	-0.0037363	0.000338	0.029087	0.000327	0.001908	4.07E-05
5	-0.0001997	0.000251	0.028723	0.000249	0.001833	3.15E-05
10	0.00191434	0.000158	0.021847	0.000132	0.001059	1.35E-05
25	0.00213479	5.28E-05	0.008783	3.65E-05	0.000175	1.49E-06
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	0.0444016	0.002236	0.089973	0.002208	0.021321	0.001159
5	0.00842001	0.000912	0.051893	0.000894	0.006246	0.00024
10	-0.0029581	0.000499	0.037391	0.000477	0.003156	7.92E-05
25	-0.0052196	0.000295	0.030218	0.000263	0.002058	3.25E-05

Table 22: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 1 and η=1000, Sample Size n=100

	1st Percentile, Sample size n=100,Beta=1					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	6.74063	0.01805	3.29454	0.011959	69.3577	0.373073
5	6.45663	0.018352	3.33817	0.013231	66.2537	0.357298
10	6.06907	0.020449	3.62504	0.016746	65.8211	0.380488
25	5.55476	0.023413	3.96201	0.018546	65.4893	0.264337
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	7.40944	0.167671	10.8272	0.158	320.796	10.8022
5	3.80438	0.094849	6.19429	0.089235	101.436	3.24221
10	2.42281	0.047664	4.10897	0.042615	43.6021	1.03388
25	1.96486	0.021211	3.05844	0.016404	24.5249	0.292278

Table 23: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 1 and η=1000, Sample Size n=5000

	1st Percentile, Sample size n=5000,Beta=1					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	-2.6739	0.027214	4.24765	0.017053	46.9457	0.249911
5	-1.81675	0.020311	3.43886	0.012556	29.37	0.178921
10	-0.95622	0.012514	2.26511	0.008719	12.2243	0.081184
25	-0.12952	0.00624	1.02831	0.003843	2.34816	0.017599
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	5.18E+00	0.22295	15.7865	0.213172	589.489	17.3587
5	1.15296	0.122004	11.3074	0.115167	286.734	6.21402
10	-0.51626	0.080917	8.85064	0.071154	174.202	2.7359
25	-0.86961	0.051262	6.9123	0.039248	106.377	1.10419

Table 24: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 1 and η=1000, Sample Size n=1000

	1st Percentile Differences (Estimated-True)					
	MLE					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	-0.009	0.021	3.265	0.012	23.507	0.174
5	0.617	0.013	2.046	0.009	9.618	0.083

10	0.694	0.003	0.481	0.002	0.993	0.009
25	1.222	0.008	1.197	0.004	4.65	0.03
	MRR					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	5.844	0.192	17.264	0.182	700.173	16.384
5	1.572	0.138	12.158	0.123	332.607	7.199
10	0.161	0.085	9.422	0.073	197.029	3.147
25	-0.323	0.039	6.473	0.033	92.597	0.934

Table 25: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 1 and η=1000, Sample Size n=5000

	1st Percentile, Sample size n=5000,Beta=1					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	-2.6739	0.027214	4.24765	0.017053	46.9457	0.249911
5	-1.81675	0.020311	3.43886	0.012556	29.37	0.178921
10	-0.95622	0.012514	2.26511	0.008719	12.2243	0.081184
25	-0.12952	0.006249	1.02831	0.003843	2.34816	0.017599
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	5.18E+00	0.22295	15.7865	0.213172	589.489	17.3587
5	1.15296	0.122004	11.3074	0.115167	286.734	6.21402
10	-0.51626	0.080917	8.85064	0.071154	174.202	2.7359
25	-0.86961	0.051262	6.9123	0.039248	106.377	1.10419

Table 26: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 3 and η=1119.95, Sample Size n=100

	1st Percentile, Sample size n=100,Beta=3					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	48.7451	0.133905	23.2996	0.084329	3572.69	18.9009
5	40.4058	0.16351	25.8394	0.099973	3105.07	20.5274
10	34.4156	0.210029	32.9232	0.123834	3574.74	24.9394
25	30.0041	0.243199	41.7317	0.14961	4739.05	31.2221
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	59.0055	0.895719	1.36E+02	0.754216	44224.5	562.874
5	34.0325	0.652352	89.5631	0.517442	18890.8	244.885
10	24.4577	0.376801	52.8638	0.241272	6767.68	72.934
25	25.0729	0.157352	31.3622	0.119055	2796.21	18.1535

Table 27: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 3 and η=1119.95, Sample Size n=1000

	1st Percentile, Sample size n=1000,Beta=3					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	20.2001	0.288241	44.2064	0.192242	4719.87	44.6326
5	12.7944	0.128052	21.889	0.091663	1220.32	9.86245
10	5.63116	0.026044	3.86125	0.017519	64.616	0.565226
25	5.50887	0.044795	8.50808	0.033892	189.911	1.31186
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	42.2794	1.18207	204.841	0.860966	94319.3	848.123
5	13.4602	0.964509	152.99	0.704925	51830.2	485.572
10	-0.43825	0.633734	108.647	0.450064	26030.9	213.568
25	-3.43156	0.426019	66.9657	0.258025	9901.41	75.6219

Table 28: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 3 and η=1119.95, Sample Size n=5000

	1st Percentile Differences (Estimated-True)					
	MLE					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	12.692	0.556	85.943	0.426	16466.6	172.44
5	2.146	0.325	55.735	0.246	6856.4	61.974
10	1.788	0.201	30.449	0.123	2048.4	17.016
25	2.346	0.062	10.565	0.038	251.6	1.838
	MRR					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	35.512	1.417	204.173	0.833	93241.7	826.87
5	9.98	1.565	155.046	1.553	53707.8	1648.76
10	-3.688	0.808	110.053	0.436	26742.1	205.98
25	-6.809	0.429	71.92	0.271	11450.3	85.72

Table 29: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 5 and η=1089.13, Sample Size n=100

	1st Percentile, Sample size n=100,Beta=5					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	68.1629	0.213518	36.7311	0.190142	7624.19	55.7681

5	53.5036	0.203537	36.3484	0.159886	5776.44	41.4127
10	40.7069	0.253706	39.7577	0.159085	5143.42	39.8713
25	33.2336	0.273728	47.4216	0.171027	6061.37	43.0622
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	63.5133	0.894996	1.69E+02	0.642038	67330.4	557.248
5	37.4984	0.683409	110.702	0.429906	28430.2	239.841
10	27.6172	0.405811	61.2611	0.219753	9038.83	73.2983
25	28.8975	0.176977	34.8213	0.138122	3507.57	21.6331

Table 30: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 5 and η=1089.13, Sample Size n=1000

	1st Percentile, Sample size n=1000,Beta=5					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	45.9748	0.382097	59.855	0.340518	10028.3	119.907
5	21.5277	0.150278	25.8109	0.116315	1933	16.4154
10	6.51794	0.028957	4.51502	0.020434	87.4666	0.75078
25	5.95198	0.048178	9.24017	0.035043	223.599	1.47297
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	44.4269	1.20075	226.667	0.753775	115175	780.68
5	14.3939	1.03019	171.652	0.658644	65172.1	502.051
10	-0.48378	0.692426	121.566	0.449432	32575.8	239.2
25	-3.77242	0.464599	74.0097	0.283437	12093	92.5476

Table 31: Average Median Differences between 1st Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β = 5 and η=1089.13, Sample Size n=5000

	1st Percentile Differences (Estimated-True)					
	MLE					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	68.1562	0.76321	123.873	0.62672	38517	427.081
5	24.6182	0.41475	68.588	0.33704	10988.4	117.873
10	5.8149	0.20392	33.605	0.1269	2523.8	20.083
25	2.5768	0.06756	11.358	0.04076	291	2.136
	MRR					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	37.1946	1.45111	226.171	0.63248	114119	656.859
5	10.5026	1.56473	172.679	0.62856	65981	554.471
10	-4.0341	0.87772	122.852	0.45286	33312	243.232

25	-7.4254	0.4685	79.443	0.30309	13969	107.655

Table 32: Average Median Differences between 10th Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β =0. 5 and η=333, Sample Size n=100

	10st Percentile Differences (Estimated-True)					
	MLE					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	1.0915	0.0223	3.3512	0.0186	26.0024	0.3137
5	1.0599	0.0119	2.0718	0.0098	10.5934	0.1068
10	0.5694	0.0027	0.3855	0.002	0.6525	0.0063
25	0.4729	0.0038	0.7113	0.0032	1.3393	0.0104
	MRR					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	1.5078	0.0681	6.6179	0.0676	99.9959	2.2715
5	0.5231	0.0476	4.9993	0.0472	55.9196	1.1227
10	0.0677	0.0266	3.78	0.0259	31.6279	0.4363
25	0.0391	0.0137	2.3927	0.0133	12.6428	0.1436

Table 33: Average Median Differences between 10th Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β =0. 5 and η=333, Sample Size n=1000

	10th Percentile, Sample size n=1000,Beta=0.5					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	-1.05671	0.036822	2.97395	0.036996	20.9896	0.426393
5	-0.454556	0.033114	3.19579	0.030148	22.9624	0.404714
10	-0.0127652	0.019617	2.8357	0.016186	17.7706	0.20293
25	0.105734	0.009865	1.55957	0.00765	5.37988	0.054254
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	1.44584	0.075306	5.59733	0.071123	72.6193	2.09413
5	0.179768	0.048625	4.13239	0.046299	38.2692	0.859788
10	-0.351443	0.028191	3.28393	0.024684	24.0396	0.346028
25	-0.41473	0.018977	2.62415	0.01601	15.4	0.174025

Table 34: Average Median Differences between 10th Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β =0. 5 and η=333, Sample Size n=5000

	10th Percentile, Sample size n=5000,Beta=0.5					
	MLE					

NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	-1.79455	0.043034	2.16322	4.27E-02	14.0872	0.296303
5	-1.05415	0.042246	2.77769	0.042012	18.6316	0.442329
10	-0.431859	0.035164	3.07668	0.032953	21.3524	0.442452
25	-0.097005	0.019298	2.50318	0.015382	13.8709	0.169252
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	1.31225	0.083665	5.18454	0.081139	62.9315	2.11886
5	0.0560351	0.047978	3.83606	0.046493	33.0473	0.824812
10	-0.418557	0.031735	3.15343	0.030057	22.3296	0.40542
25	-0.438772	0.020965	2.66015	0.017598	15.8581	0.186196

Table 35: Average Median Differences between 10^{th} Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β =1 and η=1000, Sample Size n=100

	10th Percentile, Sample size n=100,Beta=1					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	-1.16E-01	0.185113	34.7907	0.108928	2666.55	16.7837
5	6.81617	0.128241	21.9076	0.092877	1104.93	9.24203
10	7.39708	0.032844	5.15658	0.023466	113.392	0.974322
25	13.1627	0.078741	12.6791	0.047884	527.737	3.24285
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	22.2171	0.872619	116.283	0.744968	30411.8	421.958
5	6.66555	0.651965	89.8928	0.52991	17909.9	218.171
10	-0.14004	0.422277	67.371	0.29818	10013.9	88.8959
25	0.704813	0.244833	41.1967	0.164168	3742.84	30.1586

Table 36: Average Median Differences between 10^{th} Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β =1 and η=1000, Sample Size n=1000

	10th Percentile Differences (Estimated-True)					
	MLE					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	-35.538	0.318	46.716	0.197	6078.52	26.887
5	-26.024	0.27	40.026	0.133	4209.77	19.121
10	-16.104	0.177	27.954	0.097	1982.17	12.544
25	-6.382	0.089	14.984	0.055	535.71	3.694
	MRR					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	18.553	0.918	115.328	0.809	29806.1	454.569

5	0.969	0.698	90.886	0.556	18273.3	223.993
10	-4.135	0.474	71.37	0.367	11265	112.459
25	-5.785	0.282	49.224	0.182	5374.6	38.251

Table 37: Average Median Differences between 10th Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β =1 and η=333, Sample Size n=5000

	10th Percentile, Sample size n=5000,Beta=1					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	-47.2858	0.380963	48.3456	0.26706	7403.45	24.5217
5	-36.0634	0.301885	46.1752	0.159103	6001.78	22.0823
10	-24.3809	0.214441	35.2775	0.117652	3337.54	20.1638
25	-13.7589	0.123664	21.3324	0.075993	1192.37	8.79899
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	16.8902	1.02515	115.224	0.918638	29756	498.461
5	0.620057	0.67337	91.9827	0.562655	18711.9	231.792
10	-6.28438	0.478458	70.6797	0.399516	11077.8	120.833
25	-6.52682	0.319484	49.2637	0.213363	5397.21	43.1664

Table 38: Average Median Differences between 10th Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β =3 and η=1119.95, Sample Size n=100

	10th Percentile, Sample size n=100,Beta=3					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	45.7493	0.463575	99.5564	0.346427	23926.8	170.991
5	29.383	0.283885	49.3225	0.191441	6226.65	47.6053
10	12.8843	0.054079	8.73309	0.037365	334.241	2.6837
25	12.5138	0.118398	19.0359	0.07161	955.618	6.66412
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	36.6936	1.32391	251.332	0.954566	140568	1087.1
5	11.5295	1.12509	191.158	0.659566	80674.6	559.213
10	-0.316577	0.764813	134.112	0.439321	39635.1	258.053
25	1.2009	0.438628	76.1407	0.244794	12776.8	82.225

Table 39: Average Median Differences between 10th Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β =3 and η=1119.95, Sample Size n=1000

	10th Percentile, Sample size n=1000,Beta=3

	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	19.1445	1.58891	222.999	1.30365	110295	1355.89
5	-2.90637	0.952446	152.725	0.685172	51470.8	457.703
10	-0.925947	0.47565	90.387	0.310713	18002.4	123.812
25	2.1317	0.232259	39.1653	0.148847	3386.43	25.9071
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	29.7664	1.39119	253.278	0.885773	142256	1.01E+03
5	3.65471	1.15935	194.287	0.729579	83235.3	615.907
10	-9.31515	0.759853	139.958	0.472815	43249.7	289.112
25	-10.8794	0.51094	89.8396	0.321371	17907.9	126.368

Table 40: Average Median Differences between 10th Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β =3 and η=1119.95, Sample Size n=5000

	10th Percentile Differences (Estimated-True)					
	MLE					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	13.668	2.12	295.3	1.722	192956	2315.69
5	-17.994	1.315	212.892	1	100337	917.5
10	-12.402	1	140.702	0.635	43857	385.62
25	-3.098	0.466	73.887	0.281	12048	90.7
	MRR					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	26.327	1.592	253.32	0.771	142127	879.339
5	1.914	1.732	194.733	0.794	83792	761.322
10	-9.554	1.016	140.302	0.507	43519	310.919
25	-10.861	0.557	91.14	0.32	18429	130.254

Table 41: Average Median Differences between 10th Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β =5 and η=1089.95, Sample Size n=100

	10th Percentile, Sample size n=100,Beta=5					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	75.4373	0.47911	97.6416	0.4628	26716.6	257.603
5	35.53	0.24533	42.6116	0.17201	5265.98	42.3005
10	10.7965	0.04821	7.35734	0.03514	236.048	2.04199
25	9.73689	0.08503	15.0656	0.06035	595.225	4.36141
	MRR					

NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	28.7441	1.02728	197.068	0.70605	86404.2	612.28
5	9.23885	0.78224	153.071	0.54568	51713.9	367.998
10	1.76463	0.62805	107.893	0.37038	25659.8	176.398
25	0.62288	0.37846	60.7785	0.2034	8143.43	54.234

Table 42: Average Median Differences between 10th Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β =5 and η=1089.95, Sample Size n=1000

	10th Percentile, Sample size n=1000,Beta=5					
	MLE					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	124.249	1.78203	246.521	1.64385	149925	2207
5	45.9094	0.93656	142.297	0.67816	46802.5	498.128
10	7.99923	0.48095	74.1634	0.29449	12195.8	100.656
25	2.29153	0.17745	30.9305	0.11859	2114.34	16.3101
	MRR					
NF	Diff	SimErr	MAD	SimErr	MSE	SimERR
3	23.1447	1.07325	198.075	0.72277	8.70E+04	624.787
5	0.36179	1.1257	156.444	0.57731	53996	396.283
10	-8.2918	0.64023	112.471	0.40379	27950.2	200.116
25	-8.14375	0.40762	72.3435	0.27472	11603.2	89.0038

Table 43: Average Median Differences between 10th Percentile Estimate and the True Value, Average MAD and Average pseudo-MSE calculated on the Differences, for β =5 and η=1089.95, Sample Size n=5000

	10th Percentile Differences (Estimated-True)					
	MLE					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	170.95	2.27661	351.22	2.00265	301758	3805.27
5	55.628	1.31261	208.499	1.05711	99064	1103.48
10	3.879	0.78527	117.546	0.44817	30491	240.69
25	-2.212	0.36756	58.738	0.22679	7613	58.54
	MRR					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	20.463	1.22973	199.143	0.61486	87823.3	530.15
5	1.4353	1.33415	156.666	1.5706	54666	1605.28
10	-7.5839	0.80632	111.981	0.40564	27721.4	201.75
25	-8.6021	0.44168	72.648	0.2628	11709.3	85.84

4. Simulation Study for Risk Management: Prediction of Future Failures

This study uses the difference between the predicted and true number of failed units during the following year, defined to be 500 cycles/year as the evaluation criteria for the prediction of future failures. For the prediction study, we are considering the following combinations β and η with a sample size of 1000 units and 3, 5, 10 and 25 failures.

Table 44: Parameter Values for Risk Management Study

β	η
0.5	1,000,000
1.0	100,000
3.0	5,000
5.0	5,000

We use the difference between the predicted and true number of failures to evaluate how well the joint estimates of the parameters predict the number of failures over the next year. The predicted number of failures for the next year given the time (*t*) of the last failure upon which we base the estimation is:

$$E(No.Failures) = (n - r)p$$

where, n is the sample size, for this study n = 1000, r is the number of failures that have already occurred and p is the probability of failure in the next year given by:

$$p = \frac{\left[exp\left[-\left(\frac{t}{\hat{\eta}_t}\right)^{\hat{\beta}_t}\right] - exp\left[-\left(\frac{t+500}{\hat{\eta}_t}\right)^{\hat{\beta}_t}\right]\right]}{exp\left[-\left(\frac{t}{\hat{\eta}_t}\right)^{\hat{\beta}_t}\right]}$$

where *t* is the time of 3rd, 5th, 10th, or 25th failure and $\hat{\beta}_t$ and $\hat{\eta}_t$ are the estimated values for β and η based on the number of failures that have already occurred at time *t*. The true number of failures is simply the number of failures that occur in the sample of 1000 units within the interval: $[t,\ t+500]$. The quantity of interest in the prediction study is therefore:

$$D = Pred(No.Failures) - True(No.Failures).$$

The prediction study is carried out using the same two-stage simulation process as the estimation study. The two-stage process is again necessary because the differences in the prediction study are skewed similarly to the estimates of β and η. The first stage produces the estimate of the median difference over 1000 iterations. The second stage generates the sampling distribution of these differences using 100 iterations.

We then calculate the following quantities to evaluate how well the two methods predict the true number of failures in the next year.

1. The overall difference: $\bar{\tilde{D}} = \frac{\sum_{i=1}^{100} \tilde{\tilde{D}}_i}{100} = \frac{\sum_{i=1}^{100} Med\left(\tilde{D}\right)_i}{100}$
2. The simulation error of the difference: $SimError(\bar{\tilde{D}}) = \sqrt{\frac{Var(\tilde{\tilde{D}})}{100}}$
3. The average Median Absolute Deviation (MAD) of differences: $\overline{MAD}_{\mathbf{D}} = \frac{\sum_{i=1}^{100} Med\left(\left|\tilde{D} - \tilde{\tilde{\mathbf{D}}}\right|\right)_i}{100}$
4. The average pseudo-MSE of differences:

$$\overline{pseudo - MSE}_{\mathbf{D}} = \frac{\sum_{i=1}^{100}\left(\left(Med\left(\left|\tilde{D} - \mathbf{D}\right|\right)\right)^2 + (k * MAD_{\mathbf{D}})^2\right)_i}{100}$$

- Note that simulation error is the standard error of the median difference.
- We calculate a simulation error for each of the average median difference estimate, MAD, and pseudo-MSE.

The simulation results for the prediction study are displayed in Tables 45 – 48.

Table 45: Average median differences between predicted and actual number of failures to occur next year for β = 0.5

	Number of Failures Differences (Estimated-True)					
	MLE					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	1.799	0.116	14.586	6.842	474.408	6.842
5	0.745	0.063	9.853	2.357	214.915	2.357
10	0.512	0.033	5.373	0.638	63.985	0.638
25	0.405	0.014	2.425	0.084	13.124	0.084
	MRR					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	-5.2	0.159	16.315	0.142	619.015	8.767
5	-0.761	0.164	16.89	0.135	634.234	9.816

10	0.726	0.089	12.121	0.064	325.133	3.547
25	0.717	0.026	4.867	0.02	52.742	0.443

Table 46: Average median differences between predicted and actual number of failures to occur next year for β = 1.0

	Number of Failures Differences (Estimated-True)					
	MLE					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	1.4802	0.0159	2.9593	0.0194	21.5472	0.2883
5	1.2882	0.013	2.3541	0.0114	13.8858	0.1375
10	0.9927	0.0104	1.9098	0.0076	9.0258	0.0679
25	0.6466	0.0099	1.6187	0.0053	6.1934	0.0362
	MRR					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	-1.0745	0.0341	5.0301	0.0371	57.1851	0.7576
5	-0.1041	0.0456	5.4538	0.0388	65.9252	0.926
10	0.4604	0.0324	4.6918	0.022	48.8083	0.4817
25	0.5294	0.0165	3.0751	0.0116	21.1217	0.1603

Table 47: Average median differences between predicted and actual number of failures to occur next year for β = 3.0

	Number of Failures Differences (Estimated-True)					
	MLE					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	-0.261	0.039	7.108	0.042	111.672	1.332
5	0.511	0.045	7.979	0.039	140.715	1.4
10	0.819	0.051	8.428	0.039	157.379	1.499
25	0.522	0.046	8.667	0.03	165.777	1.136
	MRR					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	-2.944	0.075	10.742	0.075	264.092	3.193
5	-0.948	0.127	14.341	0.099	456.68	6.106
10	0.732	0.124	17.365	0.08	666.276	6.241
25	1.696	0.094	17.077	0.056	645.463	4.33

Table 48: Average median differences between predicted and actual number of failures to occur next year for β = 5.0

	Number of Failures Differences (Estimated-True)					
	MLE					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	-1.803	0.027	4.581	0.023	49.559	0.477

5	-1.356	0.036	6.132	0.028	84.795	0.748
10	-0.058	0.043	7.92	0.034	138.333	1.202
25	0.542	0.055	9.704	0.036	207.866	1.569
	MRR					
NF	Difference	SimErr	MAD	SimErr	MSE	SimERR
3	-1.916	0.052	8.986	0.055	182.097	2.006
5	-0.766	0.106	12.529	0.084	348.31	4.565
10	0.692	0.119	16.769	0.074	621.205	5.622
25	1.76	0.1	18.697	0.06	773.29	5.036

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